Supplemental Material for: Routing entanglement through quantum networks

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(Dated: November 13, 2025)

I. THE DYNAMICAL AND NOISE MATRIX IN THE LYAPUNOV EQUATION

The unidirectionally coupled nodes described in [1] were used as a basis for the model in this study and can be summarized through the master equation in Eq. (2) of the main text. The equivalent quantum Langevin equations for the model in this study are given by

$$\begin{aligned}\dot{\hat{a}}_k = &-\left(i\omega_k - \frac{\gamma_{\text{out}} + (2-2\delta_{k0}-\delta_{k1}-\delta_{kM})\gamma}{2}\right)\hat{a}_k \\ &+ \delta_{k0}(-r\hat{a}_0^\dagger + J_0\hat{a}_1) - \delta_{k1}J_0^*\hat{a}_0 \\ &- (1-\delta_{k0}-\delta_{k1})\gamma\hat{a}_{k-1} + (1-\delta_{k0}-\delta_{kM})\sqrt{\gamma}\hat{b}^{\text{in}}_{k,k+1} \\ &+ (1-\delta_{k0}-\delta_{k1})\sqrt{\gamma}\hat{b}^{\text{in}}_{k-1,k} + \sqrt{\gamma_{\text{out}}}\hat{b}^{\text{in}}_k \end{aligned} \tag{1}$$

in the case of connecting the source node to the first node in the passing direction, where $iJ_0 = j_{\text{fwd}}$, $J_M = 0$, and $J_1 = J_2 = ... = J_{M-1} = \gamma/2$ according to the condition required for unidirectionality. Similarly, in the case of connecting the source node to the final node to obtain a connection against the passing direction with $J_0 = 0$, $-iJ_M = j_{\text{bwd}}$ and $J_1 = J_2 = ... = J_{M-1} = \gamma/2$ the resulting dynamics turn out to be

$$\begin{aligned}\dot{\hat{a}}_k = &-\left(i\omega_k - \frac{\gamma_{\text{out}} + (2-2\delta_{k0}-\delta_{k1}-\delta_{kM})\gamma}{2}\right)\hat{a}_k \\ &+ \delta_{k0}(-r\hat{a}_0^\dagger - J_M\hat{a}_M) + \delta_{kM}J_M^*\hat{a}_0 \\ &- (1-\delta_{k0}-\delta_{k1})\gamma\hat{a}_{k-1} + (1-\delta_{k0}-\delta_{kM})\sqrt{\gamma}\hat{b}^{\text{in}}_{k,k+1} \\ &+ (1-\delta_{k0}-\delta_{k1})\sqrt{\gamma}\hat{b}^{\text{in}}_{k-1,k} + \sqrt{\gamma_{\text{out}}}\hat{b}^{\text{in}}_k \end{aligned} \tag{2}$$

The quantum Langevin equations define the dynamics of the system and can be rewritten in terms of the two quadratures $\hat{x}_k = (\hat{a}_k^\dagger + \hat{a}_k)/\sqrt{2}$ and $\hat{p}_k = i(\hat{a}_k^\dagger - \hat{a}_k)\sqrt{2}$ of each mode as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{r}(t) = \mathcal{A}\vec{r}_l(t) + \mathcal{B}\vec{r}_{\text{in}}(t) \tag{3}$$

with the quadratures arranged as the vector $\vec{r} = (\hat{x}_{\hat{a}_0}, \hat{p}_{\hat{a}_0}, \ldots, \hat{x}_{\hat{a}_M}, \hat{p}_{\hat{a}_M})^T$ and the noise vector $\vec{r}_{\text{in}} = (\hat{x}_{\hat{b}^{\text{in}}_0}, \hat{p}_{\hat{b}^{\text{in}}_0}, \ldots, \hat{x}_{\hat{b}^{\text{in}}_M}, \hat{p}_{\hat{b}^{\text{in}}_M}, \hat{x}_{\hat{b}^{\text{in}}_{1,2}}, \hat{p}_{\hat{b}^{\text{in}}_{1,2}}, \ldots, \hat{x}_{\hat{b}^{\text{in}}_{M-1,M}}, \hat{p}_{\hat{b}^{\text{in}}_{M-1,M}})^T$ where $\mathcal{A}_{kl}$ denotes a $2M \times 2M$ matrix called the dynamical matrix.

From the equations (S1) and (S2) we extract dynamical matrix $\mathcal{A}$ as it only encompasses terms containing the nodes $\hat{a}$. The dynamical matrix can be written with the 2×2 identity matrix $\mathbb{1}_2$, the 2×2 zero matrix 0_2, and the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

as

$$\mathcal{A} = \begin{bmatrix} -r\sigma_z - \frac{\gamma_0}{2}\mathbb{1}_2 + i\omega_0\sigma_y & ij_{\text{fwd}}\sigma_y & 0_2 & \ldots & ij_{\text{bwd}}\sigma_y \\ ij_{\text{fwd}}\sigma_y & -\frac{\gamma_1}{2}\mathbb{1}_2 + i\omega_1\sigma_y & 0_2 & \ldots & 0_2 \\ 0_2 & -\gamma\mathbb{1}_2 & -\frac{\gamma_2}{2}\mathbb{1}_2 + i\omega_2\sigma_y & \ldots & 0_2 \\ 0_2 & 0_2 & -\gamma\mathbb{1}_2 & \ldots & 0_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ij_{\text{bwd}}\sigma_y & 0_2 & 0_2 & \ldots & -\frac{\gamma_M}{2}\mathbb{1}_2 + i\omega_M\sigma_y \end{bmatrix} \tag{5}$$

with $\gamma_0 = \gamma_{\text{out}}$, $\gamma_1 = \gamma_M = \gamma_{\text{out}} + \gamma$ and all other $\gamma_k = \gamma_{\text{out}} + 2\gamma$ according to the number of baths to which these are connected. We can extract $\mathcal{B}$ from Eq. (1) and Eq. (2) as 2×2 identity matrices $\mathbb{1}_2$ if the respective bath is connected to the oscillator scaled by the square root of the decay rate and zeros otherwise

$$\mathcal{B} = \begin{bmatrix} \sqrt{\gamma_{\text{out}}}\mathbb{1}_2 & 0_2 & 0_2 & \dots & 0_2 & 0_2 & 0_2 & \dots & 0_2 \\ 0_2 & \sqrt{\gamma_{\text{out}}}\mathbb{1}_2 & 0_2 & \dots & 0_2 & 0_2 & \sqrt{\gamma}\mathbb{1}_2 & \dots & 0_2 \\ 0_2 & 0_2 & \sqrt{\gamma_{\text{out}}}\mathbb{1}_2 & \dots & 0_2 & 0_2 & \sqrt{\gamma}\mathbb{1}_2 & \dots & 0_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & 0_2 & \dots & \sqrt{\gamma_{\text{out}}}\mathbb{1}_2 & 0_2 & 0_2 & \dots & \sqrt{\gamma}\mathbb{1}_2 \\ 0_2 & 0_2 & 0_2 & \dots & 0_2 & \sqrt{\gamma_{\text{out}}}\mathbb{1}_2 & 0_2 & \dots & \sqrt{\gamma}\mathbb{1}_2 \end{bmatrix} \tag{6}$$

With the covariance matrix defined as

$$\mathcal{V}_{ij} = \langle \vec{r}_i \vec{r}_j + \vec{r}_j \vec{r}_i \rangle - 2\langle \vec{r}_i \rangle \langle \vec{r}_j \rangle \tag{7}$$

Through some manipulation the dynamics can be worked out in terms of the covariance matrix whereby the resulting equation is termed the Lyapunov equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V} = \mathcal{V}\mathcal{A}^T + \mathcal{A}\mathcal{V} + \mathcal{N} \tag{8}$$

with the noise matrix

$$\mathcal{N}_{ij} = \sum_q (\bar{N}_{\lfloor q/2 \rfloor} + 1/2)(\mathcal{B}_{iq}\mathcal{B}^T_{qj} + \mathcal{B}_{jq}\mathcal{B}^T_{qi}) \tag{9}$$

consisting of 2×2 blocks $\gamma_q(\bar{N}_q + 1/2)\mathbb{1}_2$ if oscillator i and oscillator j are coupled to the bath referred to in the qth index in $\vec{r}_{\text{in}}$. The concrete form of the noise matrix is

$$\mathcal{N} = \begin{bmatrix} \gamma_{\text{out}}\bar{E}_0\mathbb{1}_2 & 0_2 & 0_2 & 0_2 & \dots & 0_2 \\ 0_2 & [\gamma_{\text{out}}\bar{E}_1 + \gamma\bar{E}_{1,2}]\mathbb{1}_2 & \gamma\bar{E}_{1,2}\mathbb{1}_2 & 0_2 & \dots & 0_2 \\ 0_2 & \gamma\bar{E}_{1,2}\mathbb{1}_2 & [\gamma_{\text{out}}\bar{E}_2 + \gamma\bar{E}_{1,2} + \gamma\bar{E}_{2,3}]\mathbb{1}_2 & \gamma\bar{E}_{2,3}\mathbb{1}_2 & \dots & 0_2 \\ 0_2 & 0_2 & \gamma\bar{E}_{2,3}\mathbb{1}_2 & [\gamma_{\text{out}}\bar{E}_3 + \gamma\bar{E}_{2,3} + \gamma\bar{E}_{3,4}]\mathbb{1}_2 & \dots & 0_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & 0_2 & 0_2 & \dots & [\gamma_{\text{out}}\bar{E}_M + \gamma\bar{E}_{M-1,M}]\mathbb{1}_2 \end{bmatrix} \tag{10}$$

with $\bar{E}_q = (\bar{N}_q + 1/2)$ and $\bar{E}_{n,q} = (\bar{N}_{n,q} + 1/2)$.

II. STABILITY AND PHYSICALITY ANALYSIS OF THE SYSTEM

We have established that the covariance matrix evolves under the differential Lyapunov equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(t) =& \mathcal{A}\mathcal{V}(t) + \mathcal{V}(t)\mathcal{A}^T + \mathcal{N} \\ \mathcal{V}(0) =& \mathcal{V}_0 \end{aligned} \tag{11}$$

where we assume that $\mathcal{A}$ is diagonalizable such that $P^{-1}\mathcal{A}P = D$ with the diagonal matrix D. Consequently, the transpose is $\mathcal{A}^T$ is diagonalizable as $Q^{-1}\mathcal{A}Q = D = P^T\mathcal{A}^T(P^{-1})^T = D^T = D$ with $Q = (P^T)^{-1}$. Then, it can be shown that the solution to the dynamical Lyapunov equation is unique and can be spectrally decomposed [2]

$$\begin{aligned} \mathcal{V}(t) =& P\left[\left(\int_0^t e^{(\alpha_j+\alpha_k)(t-s)}\mathrm{d}s\right)_{jk} \circ (P^{-1}\mathcal{N}(Q^\dagger)^{-1})_{jk}\right]Q^\dagger \\ &+ P[(e^{(\alpha_j+\alpha_k)t})_{jk} \circ (P^{-1}\mathcal{V}_0(Q^\dagger)^{-1})_{jk}]Q^\dagger, \end{aligned} \tag{12}$$

where $\circ$ denotes the Hadamard product or equivalently entrywise multiplication of the matrices. Through the scalar integrals [2]

$$\int_0^t e^{(t-s)(\alpha_j+\alpha_k)}\mathrm{d}s = \begin{cases} \frac{e^{(\alpha_j+\alpha_k)t}-1}{(\alpha_j+\alpha_k)} & \text{if } (\alpha_j+\alpha_k) \neq 0 \\ t & \text{if } (\alpha_j+\alpha_k) = 0 \end{cases} \tag{13}$$

we see, that $\mathcal{V}$ acquires a unique steady state if all eigenvalues of $\mathcal{A}$ have negative real part. Then, the steady state for $t \to \infty$ is given by the first line of Eq. (12) as the second line vanishes.

A test for physicality of the resulting covariance matrix consists in satisfying two requirements [3]. Firstly, the steady state covariance matrix must be positive which means that the real parts of the eigenvalues of the covariance matrix are all equal to or greater than zero. Secondly, all symplectic eigenvalues of the covariance matrix must have real parts greater than or equal to $1/2$. The symplectic eigenvalues can be computed with the symplectic form

$$\Omega = \bigoplus_{i=0}^{M} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{14}$$

as the eigenvalues of the matrix $|i\Omega\mathcal{V}|$. The results for the

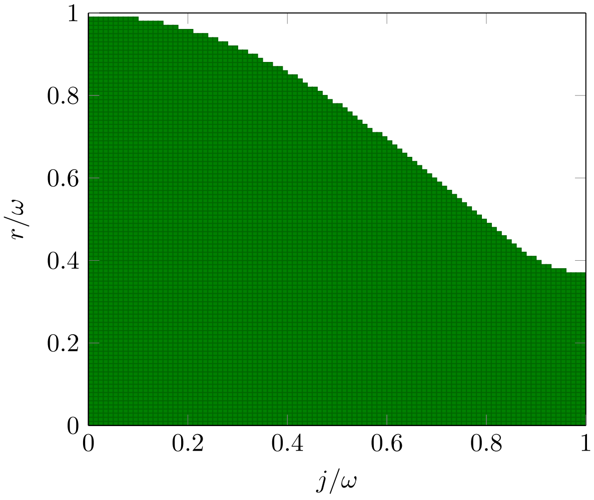

FIG. S1. Stability and physicality of the system depending on the coupling strength j/ω and squeezing parameter r/ω for the parameters $\gamma_{\text{out}}/\omega$=0.002 and γ/ω=0.8 used in the main text.

physicality happen to coincide with those of stability throughout our analyses. Therefore, the resulting stability and physicality of the system depending on the coupling strength j/ω and squeezing parameter r/ω for the parameters used in the main text are illustrated in Fig. S1.

III. DETERMINATION OF THE OPERATIONAL PARAMETERS

The matching between the cascaded quantum system with an optomechanical system as well as the aims of our analyses influence the various parameters of the model. We assumed that the mean number of excitations $\bar{N}_k = [exp(\hbar\omega/k_B T_k) - 1]^{-1}$ is chosen to observe the dynamics of the optomechanical system without the influence of noise. Thus, we set $\bar{N}_k = 0$ throughout. The resulting mean number of excitations in the system, depicted in Fig. 3(b) of the main text, can be converted into an effective temperature T_{eff} generated through the process depending on the chosen frequency ω through inversion of the above equation as

$$T_{\text{eff}} = \frac{\hbar\omega}{k_B \ln\left(1 + \frac{1}{\bar{n}_{M_{\max}}}\right)} \tag{15}$$

Using the frequency $\omega_{\text{cav}} = 2\pi \times 5$ GHz of the cavity suggested in the main text and a cut-off mean occupation of $\bar{n}_{M_{\max}} = 0.01$, we obtain an effective temperature of 52 mK that can be generated through the process which is much larger than the ambient temperature of 10 mK at which the suggested microwave resonators can be operated [4]. Thus, the effects of the non-reciprocal entanglement routing should be observable in experiments and not be hidden by thermal noise.

The values assigned for both squeezing and hopping terms, r and j respectively, are assumed to be in the small coupling region in which perturbative approaches apply, which limits these parameters to be bounded by the oscillator frequencies. Therefore, we can scale r and j accordingly in terms of ω and perform our analyses for $0 \leq r/\omega, j/\omega \leq 1$. Similarly to the temperature, the squeezing parameter r employed here is always scaled to the frequency ω and thus the squeezing to be employed in an experimental realization depends on the frequency of the system under consideration.

The value of γ was decided upon, from a bounded region identical to that of r and j, after several observations throughout the study. The chosen value for $\gamma/\omega = 0.8$ is observed to fall within a substantially narrow range where manifestation of entanglement is favoured. The coupling strength to the respective baths, $\gamma_{\text{out}}/\omega$, was observed to have a minimal impact on the system and thus its value was kept quite small, concretely $\gamma_{\text{out}}/\omega = 0.002$, so that the study would be able to focus on the more critical parameters.

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